WEYMOUTH
AM
1931
wey
upstairs
378.744
BO
AM.1931

Boston University
College of Liberal Arts
Library

THE GIFT OF the Author

378744
50
A.M. 1931
wey
1931

p.7008

BOSTON UNIVERSITY

GRADUATE SCHOOL

THESIS

THE BINOMIAL SERIES

SUBMITTED BY

ROYAL SAMUEL WEYMOUTH

A.B. BOSTON UNIVERSITY , 1928

In PARTIAL FULFILMENT OF

REQUIREMENTS FOR

THE DEGREE OF

MASTER OF ARTS

1931

BOSTON UNIVERSITY
COLLEGE OF LIBERAL ARTS
LIBRARY

p7008

378.744
BO
A.M.1931

THE BINOMIAL SERIES

OUTLINE

I

page

THE CONVERGENCE OF THE BINOMIAL SERIES. ***** I

A. GENERAL DISCUSSION. * * * * * * * * * * * * * * * * * ** I

B. ANALYSIS AND PROOFS OF CONVERGENCE AND DIVERGENCE. * * ** 2

1) WHEN $|x| < 1$, THE B.S. CONVERGES. * * * * * * * * * ** 2

2) WHEN $|x| > 1$, THE B.S. DIVERGES. * * * * * * * * * ** 2

3) WHEN $x = 1$., THE CONVERGENCE OF THE B.S. DEPENDS UPON THE VALUE OF m. * * * * * * * * * * * * * * *** 3

(a) If $m > 0$, THE B.S. CONVERGES ABSOLUTELY. ** ** 3

(b) IF $(-1 < m < 0)$, THE B.S. CONVERGES * * * * * ** 3

(c) IF $m = 0$, all terms vanish except the first. * 4

(d) Proof that for $-1 < m < 0$, $\lim_{n \to \infty} F_n = 0$. * * * * * * 4

(e) IF $m = -1$, THE B.S. IS NON-CONVERGENT. * * * * 5

(f) IF $m < -1$, THE B.S. DIVERGES. ***** * * * * ** 5

4) IF $x = -1$, THE CONVERGENCE OF THE B.S. DEPENDS UPON THE VALUE OF m, * * * * * * * * * * * * * * ** 6

(a) IF $m > 0$, THE B.S. CONVERGES ABSOLUTELY. ** * ** 6

(b) IF $M = 0$, Allterms vanish exceptthe first. * ** 6

(c) IF $m < 0$, THE B.S. IS DIVERGENT. * * * * * *** 6

II

VALUE OF THE BINOMIAL SERIES. * * * * * * * ** 7

A. IF $|x| < 1$, THE B.S. $= (1+x)^m$. * * * * * ** * * * * * ** 7

B. APPROXIMATE VALUE OF THE B.S. -LIMIT OF ERROR. * * * ** 7

C. REMARKS ON THE CONTINUITY, UNIFORM CONVERGENCE, INTEGRATION AND DIFFERENTIATION OF THE BINOMIAL SERIES. * * ** * ** 8

APPLICATIONS OF THE BINOMIAL SERIES,* * * * * * * * * ** IO

V Appendix.* * * * * * * * * * * ** II

A. Test-Ratio Test (D'Alembert's) for convergence. * * * * ** I3

B. Test-Ratio Test (D'Alembert's) for divergence. * * * * ** I3

B. Note ;* ** * * ** I3

c. Raabe's Test - for positive series. * * * * * * * ** * * ** I4

D. Definition of convergence. * * * * ** * * * * * * ** * * ** ''

VI Summary of B.S. - Convergence and value.* ** I5

VII Bibliography.* * * * * * * * * * * ** ** I6

I

THE CONVERGENCE OF THE BINOMIAL SERIES.

A. GENERAL DISCUSSION

If $f_{(n)} = \frac{m(m-1)(m-2)\cdots(m-n+2)}{(n-1)!} x^{n-1}$ where n is a positive integer,

then $1 + \sum_{n=2}^{\infty} f_{(n)} = 1 + \frac{m}{1}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)\cdots(m-n+2)}{(n-1)!}x^{n-1}$

is called the binomial series.

The value of the binomial series (abbreviated as B.S. from here on) depends upon the values of m and x . If m equals a positive integer, N', the term where $n=N'+2$ or $N'=n-2$ will contain the factor $m-n+2=N'-N'=0$ and so will vanish; so will all succeeding terms, for they will each contain the zero factor. The remaining $m+1$ terms will make up a finite polynomial of value $(1+x)^m$ discussed in high school algebra under the title "Binomial Theorem".

If m is not a positive integer, let $(m-N''+1)$ be a general expression for each numerical factor of the numerator of the n^{th} term, where N" assumes successive values $(1,2,\ldots\ldots(n-1))$. Then in no case can a factor of the n^{th} term equal zero, for this would require that $m-N''+1=0$ or $m=N''-1$, a positive integer. Then, unless $x=0$, the series has an infinite number of terms.

If $\lim_{n\to\infty} 1 + \sum_{2}^{\infty} f_{(n)}$ equals a finite number, the B.S. is said to converge , otherwise , to diverge. See def. of convergence, P. 13 14

B.ANALYSIS AND PROOFS OF CONVERGENCE AND DIVERGENCE.

I) WHEN $|x| < I$, THE B.S. CONVERGES.

m is assumed to have any real value except zero or a positive integer.

consider the series of absolute values of the terms of the B.S.

$$|I| + |mx| + \left|\frac{m(m-I)}{2!}x^2\right| + \cdots + \left|\frac{(m-I)\cdots(m-n+2)}{(n-I)!}x^{n-1}\right| + \cdots$$

The Test-ratio Test says:

If $\lim_{n\to\infty}\left|\frac{f(n+1)}{f(n)}\right|$ exists, $= \rho$, the series is:

convergent when $\rho < I$,

divergent when $\rho > I$.

$$\lim_{n\to\infty}\left|\frac{f(n+1)}{f(n)}\right| = \lim_{n\to\infty}\left|\frac{m(m-I)\cdots(m-n+I)x^n}{n!}\cdot\frac{(n-I)!}{m(m-I)\cdots(m-n+2)x^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(m-n+I)}{(n)}x\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{m}{n}-I+\frac{I}{n}\right)x\right| = |-Ix| = |x| = \rho.$$

therefore the series of absolute values converges and the B.S. is absolutely convergent when $|x| = \rho < I$.

2) WHEN $|x| > I$, THE B.S. DIVERGES.

The series of absolute values diverges, for $|x| = \rho > I$

The B.S. cannot be convergent because it would either eventually, after, say, m terms, become the absolute value series or its negative, or it would be alternating. In that case it would diverge. See Test-ratio Test -note-page 13

3) WHEN x=I, THE CONVERGENCE OF THE B.S. DEPENDS UPON THE VALUE OF m.

When x=I,, the B.S.is:

$$I+m+\frac{m(m-I)}{2!}+\frac{m(m-I)(m-2)}{3!}+\ldots+\frac{m(m-I)\ldots(m-n+2)}{(n-I)!}+\ldots$$

and $F_{(N)}=\frac{m(m-I)\ldots(m-n+2)}{n!}$, where $f_{(n)}=F_{(n)}$ when $|x|=1$.

For $n>m+2$, the B.S. alternates in sign:

Considering the absolute value series, $(n>m+2)$.

$$A=\sum_{m+3}^{\infty}F_{(N)}=\sum_{m+3}^{\infty}\left|\frac{m(m-I)(m-2)\ldots\ldots\ldots\ldots(m-n+2)}{(n-I)}\right|, \text{ where}$$

$$A_N=\sum_{m+3}^{N}\left|\frac{m(m-I)(m-2)\ldots\ldots\ldots\ldots(m-n+2)}{(n-I)!}\right|$$

To apply Raabe's Test - see appendix - where:

$$\left|\frac{f_{(n)}}{f_{(n+1)}}\right|=\left|\frac{n}{m-n+I}\right|=\left|-\frac{n}{n-m-I}\right|=\frac{n}{n-m-I}$$

$$R_{(n)}=n\left(\left|\frac{f_{(n)}}{f_{(n+1)}}\right|-I\right)=n\left(\frac{m+I}{n-m-I}\right)=\frac{m+I}{I-\frac{m+I}{n}}$$

$$R=\lim_{n\to\infty}R_{(n)}=\lim_{n\to\infty}\frac{m+I}{I-\frac{m+I}{n}}=m+I\ .$$

(a) IF $m>0$, THE B.S. CONVERGES ABSOLUTELY.

If $m>0$, $R>I$, so that the series of absolute terms converges by Raabe's Test;and the B.S.is absolutely convergent.

(b) IF $(-I<m<0)$, THE B.S.CONVERGES.

If $m<0$, $R<I$, so that the absolute value series diverges, $\frac{F_{(n)}}{F_{(n+1)}}$ is negative for $n>m+I$, so the B.S.is alternating.

The terms form a decreasing sequence $\left(\text{for } \left|\frac{F_{(n+1)}}{F_{(n)}}\right|\right.$

$\left.=\left|\frac{n+I}{n}-I\right|<I, \text{for } n>m+I; \lim_{n\to\infty}F_{(n)}=0, \text{See(d)below.}\right)$

Therefore the B.S. converges.

(c) If $m=o$, all terms vanish except the first.

(d) Proof that for $-I<m<0$, $\lim_{n\to\infty} F_{(N)} = 0$:

$$F_{(N)} = \left(\frac{m}{I}\right)\left(\frac{m-I}{2}\right)\left(\frac{m-2}{3}\right)\cdots\cdots\left(\frac{m-n+2}{n-I}\right).$$

$F_{(N)}$ is a product: calling $(k+1)$ the number of the factor, K the number of the last factor, F_K the value of F_N as a product of K factors ; and $\prod$ denoting "product":

$$F_{(K)} = \prod_{k=0}^{k=K}\left(\frac{m-k}{k+I}\right) = \pm\prod_{k=0}^{k=K}\left(\frac{k-m}{k+I}\right) = \pm\prod_{k=0}^{k=K}\left(I-\frac{I+m}{k+I}\right)$$

The $\pm$ means that it is positive or negative depending upon the value of K,

$$\lim_{N\to\infty} F_{(N)} = \lim_{K\to\infty} F_K = \pm\prod_{k=0}^{\infty}\left(\frac{k-m}{k+I}\right) = \pm\prod_{K=I}^{\infty}(I-B_{(K)}), \text{ where } B_{(K)} = \frac{I+m}{I+k}.$$

here $B_{(K)}$ is fractional for $|m|<k$ and positive for $|m|<I$

We shall prove that :

$$\lim_{N\to\infty} F_{(N)} = \lim_{K\to\infty} F_{(K)} = 0$$

in three steps. For convenience in reading, the proof will extend across the entire page: also $B_K = B_K$

$$I \pm \prod_{K=I}^{K=L}(I-B_K) < (I+\sum_{K=I}^{i} B_K)^{-1}, \text{ where } i \text{ is any value of K.}$$

Proof;

$$\prod_{K=I}^{L}(I+B_K) = (I+B_1)(I+B_2)\cdots\cdots(I+B_i)$$

$$= I+B_1+B_2+\ldots+B_i+B_1B_2+B_1B_3+\ldots+B_1B_i+B_2B_3+\ldots+B_2B_i+\ldots\ldots$$
$$+B_{i-1}B_i+B_1B_2B_3+\ldots+(B_1B_2\ldots\ldots B_i).$$

$$> (I+\sum_{k=1}^{i} B_K)$$

Then:

$$\frac{I}{\prod_{K=I}^{L}(I+B_K)} < (I+\sum_{k=1}^{i} B_k)^{-1}$$

Also: $\prod_{K=1}^{i} (I-B_K) < \frac{I}{\prod_{K=1}^{i} (I+B_K)}$ for $I-B = \frac{I-B_K^2}{I+B_K} < \frac{I}{I+B_K}$

because B_K^2 being positive and fractional makes $I-B_K^2 < I$.

Therefore $\prod_{K=1}^{i} (I-B_K) < \prod_{K=1}^{i} (I+\sum_{K=1}^{i} B_K)^{-1}$.

2 $\sum_{1}^{\infty} B_K$ is divergent.

Proof:

$B_K = \frac{I+m}{k+I} = \frac{Q}{K+I} = Q\,\frac{I}{n}$, where Q is a constant.

So $\sum B_K$ is divergent as the multiple of a divergent series.

3 $\prod_{1}^{\infty} (I-B_K) = 0$

Proof:

Since B_K is divergent , we can find K, such that for any positive number, G ,however large (for $K_2 \geq K_1$):

$\sum_{K=1}^{K=K_2} B_K = B_1+B_2+ \ldots +B_{K_2} > G.$

$I+\sum_{K=1}^{K=K_2} B_K > I+G.$

$$\frac{I}{I+G} > \frac{I}{I+\sum_{K=1}^{K=K_2} B_K} > \prod_{K=1}^{K=K_2} (I-B_K)$$

Let $G \to \infty$; $\frac{I}{I+G} \to 0$; $\sum_{K=1}^{K=K_2} (I-B_K) \to \prod_{1}^{\infty} (I-B_K) = 0$.

$\prod_{1}^{\infty} (I-B_K) = \lim_{K=\infty} F_K = \lim_{N=\infty} F_N = 0$ Q.E.D.

Resuming the former indentation.

(e) IF m=-I, THE B.S. IS NON-CONVERGENT.

$$B.S. = I - I + \frac{(-I)(-2)}{I \cdot 2} + \frac{(-I)(-2)(-3)}{I \cdot 2 \cdot 3} + \ldots$$

$$= I - I + I - I + \ldots$$ { value oscillates between 0 and 1.

(f) If m<-I, the B.S. DIVERGES.

$\left|\frac{F(n+1)}{F(n)}\right| = \left|\frac{m+I}{n} - I\right| > I$ for every n ; so that the series of absolute values diverges by the Test-ratio Test, and the B.S. diverges by the note to that test see P.

(4) IF $x=-1$, THE CONVERGENCE OF THE B.S. DEPENDS UPON m.

The B.S. is $1-m+\frac{m(m-1)}{2!}-\frac{m(m-1)(m-2)}{3!}+\cdots$

$$\pm\frac{m(m-1)\cdots(m-n+2)}{(n-1)!}\mp\cdots$$

Beginning with the term where first $n>m+1$, all the terms will have the same sign.; for $\frac{F_{n+1}}{F_n}=-1\left(\frac{m-n+1}{n}\right)$, is positive.

Raabe's Test:

$$R_n=n\left(\left|\frac{f_n}{f_{n+1}}\right|-1\right)\to 1+m \quad \text{as } n\to\infty.$$

(a) IFm $m>0$, THE B.S. CONVERGES ABSOLUTELY.

for $\lim_{n\to\infty} R_n=1+m>1$

(b) IF $m=0$, All terms vanish except the first.

(c) IF $m<0$, THE B.S. IS DIVERGENT.

for $\lim_{n\to\infty} R_n=1+m<1$.

VALUE OF THE BINOMIAL SERIES

A. IF $|x| < I$, THE B.S. $= (I+x)^m$.

(Adapted from Smail) .

(The full width of the paper will be used.)

Let $F(x) = I + F_{(m)}$ = B.S.

$$F'(x) = m\left\{I + \frac{m-I}{I!}x + \frac{(m-I)(m-2)}{2!}x^2 + \ldots + \frac{(m-I)(m-2)..(m-n+I)x^{n-1}}{(n-I)!} + \cdot\right\}$$

See section D- Remarks on Integration & Differentiation of Series.

$$(I+x)F'(x) = m\left\{I + \frac{m-I}{I!}x + \frac{(m-I)(m-2)}{2!}x^2 + \ldots + \frac{(m-I)(m-2)..(m-n+I)x^{n-1}}{(n-I)!} + \ldots\right.$$
$$\left. + x + \frac{(m-I)}{I!}x^2 + \ldots + \frac{(m-I)(m-2)..(m-n+2)x^{n-1}}{(n-2)!} + \ldots\right\}.$$

$$= m\left\{I + mx + \frac{(m-I)(m-2+2)x^2}{2!} + \ldots + \frac{(m-I)(m-2).(m-n+2)(m-n+I+n-I)x^{n-1}}{(n-I)!} + \ldots\right.$$

$$= m\left\{I + mx + \frac{m(m-I)x^2}{2!} + \ldots + \frac{(m)(m-I)(m-2)..(m-n+2)}{(n-I)!}x^{n-1} + \ldots\right.$$

$(I+x)F'(x) = mF(x)$

$\frac{F'(x)}{F(x)} = \frac{m}{I+x}$; then integrating:

$\text{Log } F(x) = m \log(I+x) + \log C$

$F(x) = C(I+x)^m$. But- $F(0) = C = I$, So:

$F(x) = (I+x)^m$ = value of the B.S. when $|x| < I$. Q.E.D.

B. APPROXIMATE VALUE OF THE B.S. & LIMIT OF ERROR.

(Adapted from P102 -Smail.....Of use in the applications of the B.S. See section III.)

If in a given Positive series, F(x), we have:

$\frac{f_{n+1}}{f_n} < r < I$, and we compute the sum of the first m terms as an approximation to the sum of the series, the error in the approximation is less than $f_{(m)} \cdot \frac{r}{I-r}$.

proof:

The error is obviously:

$$f_{(m+1)} + f_{(m+2)} + \cdots\cdots$$

But since $\frac{f_{(m+1)}}{f_{(m)}} < r$, we have

$f_{(m+1)} < r f_m$

$f_{(m+2)} < r f_{(m+1)} < r^2 f_m$

. . .

. . .

error $= f_{(m+1)} + f_{(m+2)} + f_{(m+3)} \cdots \cdots < f_m r(1 + r + r^2 + \cdots\cdots)$

By elementary algebra, the parenthesis equals $\frac{1}{1-r}$

Whence the error $< \frac{f_{(m)} \cdot r}{1-r}$

In the B.S. $(1+x)^{m'}$, $\frac{f_{(n+1)}}{f_{(n)}} < |x| < 1$, if $|x| < 1$

for $\left|\frac{f_{(n+1)}}{f_{(n)}}\right| = \left|x\left(\frac{m}{n} - 1 + \frac{1}{n}\right)\right| < |x|$ for $n > m'$

whence the $|x|$ can be considered to be the r at the top of the page.

As an example:

Suppose $m' = 3.5$, $x = .5$ and 6 terms are computed so $m = 6$

$$f_m = f_6 = \frac{m(m-1)(m-2)(m-3)(m-4)}{5!} x^5$$

$$f_{(m)} \cdot \frac{r}{1-r} \cdot = \frac{(3.5)(2.5)(1.5)(.5)(-.5)(.5)^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 5}$$

.00085 , the maximum error in approximating the value of the B.S.

C. REMARKS ON THE CONTINUITY, UNIFORM CONVERGENCE , INTEGRATION AND DIFFERENTIATION OF THE BINOMIAL SERIES.

I. A series F(x) is uniformly convergent within an interval (a,b if a certain number of terms (n) of F(x) can be determined such that for all values of x in that interval, the sum of the remaining terms $|R_n(X)| < \varepsilon$

2. A power series is uniformly convergent within its interval of convergence.

3. If $f_0(x)+f_1(x)+\ldots\ldots$, $a \leq x \leq b$, is a series of continuous functions convergent at every point of this interval (a,b), then the function, $F(x)$, represented by this series will be continuous throughout this interval if a set of positive numbers , $M_0, M_1, M_2, \ldots$ independent of x can be found such that:

$$|f_n(x)| \leq M_n \text{ , } a \leq x \leq b \text{ , } n=0,1,2,3,\ldots\ldots;$$

& $M_0+M_1+M_2+\ldots$ is a convergent series.

4. A power series represents a continuous function within its interval of convergence.

5. A power series can be integrated term by term throughout any interval (a,b) contained in the interval of convergence and not reaching out to the extremities of that interval.

6. A power series can be differentiated term by term at any point within the interval of convergence.

7. Since the interval of convergence of the B.S. is $-c \leq x \leq c$ where $|c| < 1$:

The B.S. { is uniformly convergent,
is a continuous function,
can be integrated term by term,
can be differentiated term by term,
within this interval.

IV

APPLICATIONS OF THE BINOMIAL SERIES.

A. To find the value of $\left(\frac{3}{2}\right)^{3\frac{1}{2}}$

$$\left(\frac{3}{2}\right)^{3\frac{1}{2}} = (1+.5)^{\frac{7}{2}} = (1+\tfrac{1}{2})^{\frac{7}{2}}$$

Applying $(1+x)^m$, the B.S. converges, for here $x = \frac{1}{2} < 1$.

$$\text{B.S.} = 1 + \frac{7}{2}\cdot\frac{1}{2} + \frac{\frac{7}{2}\cdot\frac{5}{2}}{1\cdot 2}\left(\tfrac{1}{2}\right)^2 + \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}}{1\cdot 2\cdot 3}\left(\tfrac{1}{2}\right)^3 + \cdots\cdots$$

$$= 1 + \frac{7}{4} + \frac{35}{32} + \frac{105}{384} + \frac{35}{2048} + \frac{7}{8192} + \cdots\cdots$$

$$= 4.13343 + R_n$$

applying the limit of error theorem under section II, part B, the maximum error is found to be .00085 .

Hence $\left(\frac{3}{2}\right)^{3\frac{1}{2}}$ is between 4.13428 and 4.13258.

B. To obtain the fifth root of 35. - See Osgood P.28.

$$35 = 32\left(\frac{35}{32}\right) = 2^5\left(1+\frac{3}{32}\right)$$

$$\sqrt[5]{35} = 2\left(1+\frac{3}{32}\right)^{\frac{1}{5}}$$

$$= 1 + \frac{1\cdot 3}{5\cdot 32} + \frac{\frac{1}{5}\cdot-\frac{4}{5}}{1\cdot 2}\left(\frac{3}{32}\right)^2 + \frac{\frac{1}{5}\cdot-\frac{4}{5}\cdot-\frac{9}{5}}{1\cdot 2\cdot 3}\left(\frac{3}{32}\right)^3 + \cdots\cdots$$

$$= 1 + .018750 - .000703 + .000040 + .000003 +$$

$$= 1.019496 \quad \text{to 5 terms}$$

$$\sqrt[5]{35} = 2.038992 + R_n.$$

To find maximum error :

$$\text{m.e.} = 2f_n\left(\frac{r}{1-r}\right) = 2\left(\frac{1\cdot\frac{-4}{5}\cdot\frac{-9}{5}\cdot\frac{-14}{5}}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{3}{32}^4\cdot\frac{\frac{3}{32}}{\frac{29}{32}}\right) = .0000005$$

whence $\sqrt[5]{35}$ is 2.038992 correct to 6 decimal places.

C. Computation of π - See Osgood.

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$= 1+\tfrac{1}{2}x^2+\frac{\left(-\frac{1}{2}\cdot-\frac{3}{2}\right)}{1\cdot 2}x^4-\left(\frac{-\frac{1}{2}-\frac{3}{2}-\frac{5}{2}}{1\cdot 2\cdot 3}\right)x^6+\cdots\cdots$$

$$= 1+\tfrac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\cdots\cdots$$

$$\sin^{-1} h = \int_0^h \frac{dx}{\sqrt{1-x^2}}$$

$$= (\text{for } |h|<1)\left[x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7+\cdots\cdots\right]_0^h$$

$$h+\frac{1}{6}h^3+\frac{3}{40}h^5+\frac{5}{112}h^7+\cdots\cdots$$

See section II,D Remarks on Integration.

If $h=\frac{1}{2}$, $\sin^{-1} h = \frac{\pi}{6}$

$$\frac{\pi}{6} = \frac{1}{2}+\frac{1}{6}\left(\frac{1}{2}\right)^3+\frac{3}{40}\left(\frac{1}{2}\right)^5+\frac{5}{112}\left(\frac{1}{2}\right)^7$$

$=.522816$

$\pi = 3.136896$.

$= 3.14$ correct to 2 decimal places.

D. To compute the perimeter of an ellipse whose major axis is twice as long as its minor axis.

Let a = semi-major axis , b = semi-minor axis , a=2b .

Call $\frac{a^2-b^2}{a^2} = e^2$

Whence $\frac{4b^2-b^2}{4b^2} = \frac{3}{4} = e^2 < 1$, if $b \leq 1$.

The general equations for the ellipse ; $y = a \sin\phi$
$x = b \cos\phi$

and for the arc measured from the end of the minor axis:

$$s = a\int_0^\phi \sqrt{1-e^2\sin^2\phi}\, d\phi .$$

Let $e^2 \sin^2\varphi = x^2$

Then $\sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$

$$1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{3}{48}x^6 - \ldots\ldots \quad \text{for } |x| < 1$$

Whence $s = a\int_0^{\varphi} (1 - \frac{1}{2}e^2\sin^2\varphi - \frac{1}{8}e^4\sin^4\varphi - \frac{3}{48}e^6\sin^6\varphi \ldots.)d\varphi$

for $|x| < 1$, s equals:

$$a\left[\varphi - \frac{1}{2}e^2\int_0^{\varphi}\sin^2\varphi d\varphi - \frac{1}{8}e^4\int_0^{\varphi}\sin^4\varphi d\varphi - \frac{3}{48}e^6\int_0^{\varphi}\sin^6\varphi d\varphi \ldots\right]$$

To find $s = \frac{1}{4}$ of perimeter, let $\varphi = \frac{\pi}{2}$, & using formula 483

of Pierce's Short Table of Integrals:

$$\int_0^{\frac{\pi}{2}}\sin^n\varphi d\varphi \quad \frac{1\cdot 3\cdot 5 \ldots\ldots(n-1)}{2\cdot 4\cdot 6 \ldots\ldots\cdot(n)} \cdot \frac{\pi}{2} \quad \text{, for n an even integer.}$$

$$s = a\cdot\frac{\pi}{2}\left[1 + \left(\tfrac{1}{2}\right)^2 e^2 - \left(\tfrac{1}{2}\cdot\tfrac{3}{4}\right)^2 \frac{e^4}{3} - \left(\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\right)^2 e^6/5 - \ldots\ldots\right]$$

Substituting values $e^2 = \frac{3}{4}b^2$, $a = 2b$, $P = 4s$.

$$P = 4b\pi\left(1 - \left(\tfrac{1}{2}\right)^2\tfrac{3}{4}b^2 - \left(\tfrac{1}{2}\cdot\tfrac{3}{4}\right)^2\left(\tfrac{3}{4}\right)^2 b^4 - \left(\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\right)^2\left(\tfrac{3}{4}\right)^3\left(\tfrac{1}{5}\right)b^6 - \ldots\right)$$

Calling b = 1 unit

$$P = 4\pi\left(1 - \frac{3}{16} - \frac{27}{1024} - \frac{6075}{737280} - \ldots\right)$$

= 9.780 times the semi-minor axis, b .
= 4.890 times the semi-major axis, a .

D. Period of oscillation of a pendulum.

The formula in mechanics for the time of a complete oscillation of a pendulum of length l is given by :

$T = 4\sqrt{\frac{l}{g}}\cdot\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$, where $k = \sin\frac{\alpha}{2}$ and α equals the initial inclination of the pendulum to the vertical.

Substituting $k^2\sin^2\varphi$ for x^2 in the formula :

$$T = \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \ldots \quad \text{, and integrating:}$$

$$T = 2\pi\sqrt{\frac{l}{g}}\cdot\left(1 + \left(\tfrac{1}{2}\right)^2 k^2 + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \ldots\ldots\right)$$

If $\alpha < 5$, $k < .0436$; $k^2 < .00191$; $k^4 < .000036$; $\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{3}{8}\right)^2 k^4 < .000418 < .001$

Calling $T = 2\pi\sqrt{\frac{l}{g}}$, the error $< .001\left(2\pi\sqrt{\frac{l}{g}}\right)$

V. Appendix.

A. Test-Ratio Test (D'Alembert's) for convergence of positive series.

If $\lim \frac{f_{n+1}}{f_n}$ exists, $=\rho$, then $\sum f_n$ is convergent if $\rho < 1$.

Proof:

If $\lim \frac{f_{n+1}}{f_n} = \rho < 1$, then for $n > m$ and $\rho < r < 1$

$\frac{f_{(m+2)}}{f_{(m+1)}} < r$; $f_{(m+2)} < rf_{(m+1)}$

$\frac{f_{(m+3)}}{f_{(m+2)}} < r$; $f_{(m+3)} < rf_{(m+2)} < r^2 f_{(m+1)}$

.

.

$\frac{f_{(m+n+1)}}{f_{(m+n)}} < r$; $f_{(m+n+1)} < rf_{(m+n)} < \ldots\ldots < r^n f_{(m+1)}$.

Adding: $f_{(m+2)} + f_{(m+3)} + \ldots + f_{(m+n+1)} < (r + r^2 + \ldots r^n) f_{(m+1)} < \frac{r}{1-r} f_{(m+1)}$

Let $n \to \infty$

$\sum_1^{\infty} f_{(n)} < f_1 + f_2 + \ldots\ldots + f_{m+1} + \frac{r}{1-r} \cdot f_{(m+1)}$, a finite number, which proves the convergence.

B. Test-Ratio Test (D'Alembert's) for divergence. of positive series.

If $\lim \frac{f_{(n+1)}}{f_{(n)}}$ exists $= \rho$, then $\sum f_n$ is divergent if $\rho > 1$.

Proof:

$\frac{f_{(m+1)}}{f_{(m)}} > S$; $f_{(m+1)} > Sf_{(m)}$ where $\rho > S > 1$

.

.

.

$\frac{f_{(m+n+1)}}{f_{(m+n)}} > S$; $f_{(m+n+1)} > Sf_{(m+n)} > S^n f_{(m)}$

So $(f_{m+1} + f_{m+2} + \ldots + f_{m+n+1}) > f_m (S + S^2 + \ldots S^n)$

Let $n \to \infty$; But $(S + S^2 + S^3 + \ldots\ldots\ldots) = \infty$

Therefore $\sum_{m+1}^{\infty} f_n = \infty = \sum_1^{\infty} f_n$ and so diverges.

B-NOte ; This test applies to alternating series also , for if

$f_{(m+1)} > Sf_m$, $\lim_{n \to \infty} f_n \neq 0$, thereby not fulfilling the condition for the convergence of alternating series. So the alternating seriesis is divergent. (Oscillating series do not approach a limit.)

C. Raabe's Test - for positive series.

Given $R = \lim_{n\to\infty} n\left(\frac{f_{(n)}}{f_{(n+1)}} - 1\right)$ and $R_n = n\left(\frac{f_{(n)}}{f_{(n+1)}} - 1\right)$.

Then $\sum_1^\infty f_n$ is convergent if $R > 1$; divergent if $R < 1$

Proof: (1) If $\lim_{n\to\infty} n\left(\frac{f_{(n)}}{f_{(n+1)}} - 1\right) > 1$, then for $n > m$:

$n\left(\frac{f_{(n)}}{f_{(n+1)}} - 1\right) > K > 1$, or $n\frac{f_{(n)}}{f_{(n+1)}} - n - 1 > K - 1 > 0$

$nf_{(n)} - (n+1)f_{(n+1)} > (K-1)f_{(n+1)}$ and if we call $K-1 = K_1$:

$K_1 f_2 < f_1 - 2f_2$

$K_1 f_3 < 2f_2 - 3f_3$

.

.

.

$K_1 f_{(n+1)} < nf_{(n)} - (n+1)f_{(n+1)}$

Adding; $K_1(f_2 + f_3 + \ldots f_{(n+1)}) < f_1 - (n+1)f_{(n+1)}$

$K_1(f_1 + f_2 + \ldots f_{(n+1)}) < K_1 f_1 + f_1 - (n+1)f_{(n+1)} < K_1 f_1 + f_1$

$\sum_1^{n+1} f_{(n)} < f_1\left(1 + \frac{1}{K_1}\right)$, a finite quantity and so converges, as $n \to \infty$

(2) If $\lim_{n\to\infty} n\left(\frac{f_{(n)}}{f_{(n+1)}} - 1\right) < 1$, then $n\left(\frac{f_{(n)}}{f_{(n+1)}} - 1\right) < S < 1$, $N > m$

Let $R_n = n\left(\frac{f_n}{f_{n+1}} - 1\right) - 1 < S - 1 < 0$

Then $\frac{f_{n+1}}{f_n} = \frac{n}{R_n + n + 1} > \frac{n}{n+1}$ (for R_n is negative and its removal decreases the fraction.

Now $\lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{1}$; $\lim_{n\to\infty} \frac{f_{(n+1)}}{f_{(n)}} > \lim_{n\to\infty} \frac{n}{n+1}$; $\lim_{n\to\infty} \frac{f_{(n+1)}}{f_{(n)}} > p > 1$ where $p > n+1$.

Therefore $\sum_1^\infty f_n$ is divergent by the Test-Ratio Test.

VI Summary of B.S. - Convergence and value.

Value of x	Value of m	Binomial Series	Value of Binomial series
$x>I$ Positive integer	*****	Finite	$(I+x)^m$
$x>I$ Not a pos. int.	*****	Divergent	∞ or $(I+x)^m$
$x=I$	$m>0$	Absolutely Convergent	$2^m = (I+x)^m$
	$m=0$	Finite-One term	$2^0 = I = (I+x)^m$
	$-I<m<0$	Conditionally Convergent	$2^m = (I+X)^m$
	$m=-I$	Non-convergent	Oscillates between zero and I. Mean value is $\frac{1}{2}$. aAlso $(I+x)^m = \frac{1}{2}$.
	$m<-I$	Divergent	∞ or $(I+x)^m$
$0<x<I$	*****	Absolutely Convergent	$(I+x)^m$
$x=0$	*****Finite		$(I+x)^m = 1$
$-I<x<o$	*****	Absolutely Convergent	$(I+x)^m$
$x=-I$	$m>o$	Absolutely Convergent	$(I+x)^m$
	$m=0$	Finite	$(I+x)^m = I$
	$m<o$	Divergent	$(I+x)^m = \infty$
$x<-I$	*****	Divergent	$(I+x)^m = \infty$

Thus in nearly every case , the B.S. reduces to a function of $(I+x)^m$

Note: Section IV of thesis not included here, because it consists simply of applications.

VI . Bibliography.

A. Introduction to Infinite Series.

by William F. Osgood,Ph.D.,L.L.D.,Professor of Mathematics in Harvard University.

Third edition.

I897- Reprinted for third Edition I920.

Introduction and first two chapters studied.

B. Introduction to the theory of infinite Series.

by T.J Bromwich, M.A.,F.R.S.

Macmillan and Co. Limited , London.

I908

Chapters I,II, III and P 6I of chap.VIII studied.

C. Elements of the Theory of Infinite Processes.

by Lloyd L. Smail,Ph.D. , Assistant Professor of Mathematics in University of Oregon.

First Edition.

Mc'Graw-Hill Book Co.,Inc.,New York.

I923

The thesis as a whole is based upon this book, particularly chapters I,II,VII,VIII,Ps I07,I08, XIV, Ps 189,193,XVII Ps ,243,244,245,247,248.

BOSTON UNIVERSITY
1 1719 02574 9054